$$w = r\sqrt{\frac{2p_0}{z+1}}, \qquad \varphi = \vartheta + \arcsin\left(w_0 \frac{z-2}{\sqrt{z+1}}\right)$$

$$p = p_0 r^2, \qquad \vartheta = \varphi_0 + \frac{1}{2}\arcsin\frac{(1-2w_0^2)(z+1) - 2w_0^2}{(z+1)\sqrt{1-4w_0^2}}$$

This solution is the same as (4.4), except for the expression for pressure.

4.2.7. The last subgroup with operator $X_5 + X_6$ generates the solution $w = e^{\vartheta} W(r)$, $\varphi = \vartheta + \Phi(r)$ and $p = e^{2\vartheta} P(r)$.

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Translated by J. J. D.

UDC 533.6

PENETRATION OF A CONE INTO A COMPRESSIBLE FLUID

PMM Vol. 37, №1, 1973, pp. 84-93

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(Received March 31, 1872)

The self-similar axisymmetric problem of penetration of a blunt rigid cone into a half-space of perfect compressible fluid is considered in linear formulation.

The problem of penetration of a blunt cone into an incompressible fluid was investigated theoretically [1-4] and experimentally [3, 5]. This problem was solved for a compressible fluid in [6] on the assumption that the radius of the intersection circle between the cone and the unperturbed fluid surface inctreases, when the penetration velocity exceeds the speed of sound in the fluid (the supersonic case).

An exact analytical solution of this problem in the subsonic case is derived here with allowance for the rise of the fluid free surface in the cone neighborhood. The distribution of pressure and forces acting on the cone is presented in terms of elementary functions, and the rate of increase of the cone wetted surface radius is determined. It is shown that in the limit case of incompressible fluid the obtained results coincide with published data, while in the other limit case the derived solution coincides with that for the case of supersonic penetration. The obtained solution is compared with that known in the theory of incompressible fluid.

1. Statement of the problem. The problem of penetration of a rigid cone with vertex angle $\pi - 2\beta$ at velocity v_0 into a perfect compressible fluid which at rest occupies the half-space $z \geqslant 0$ (Fig. 1). The velocity v_0 of the cone is directed along its axis which is perpendicular to the plane $z = 0$ and is assumed to be $v_0 \ll a$ (a is the speed of sound in the fluid at rest). It is further assumed that $\beta \ll 1$ (a blunt cone) and that during its penetration into the fluid the density of the latter varies insignificantly. The subsonic case in which $v_0 \operatorname{ctg} \beta < a$ is considered. This problem is, evidently, axisymmetric and self-similar. It will be readily seen from the physical pattern of flow that the fluid free surface in the vicinity of the cone will rise during penetration, thus increasing the cone wetted surface. Hence the correct statement of the problem requires the allowance for the rise of the fluid which, owing to the smallness of angle β, can substantially increase the cone wetted surface. We denote the unknown radius of the wetted surface by ct (Fig. 1), where t is the time and ct is the distance of the point of contact of the fluid free surface from the cone axis (we neglect the effect of the spray stream, since for small β the vertical component of the spray stream momentum is small [3]). The self-similarity of the problem implies that $c =$ const. In solving the hydrodynamic problem we shall provisionally assume that c is a given value. Linearizing the equations of motion of the fluid and the boundary conditions [2], for the velocity vector $\mathbf{v} = \{v_r(t, r, z), v_z(t, r, z)\}$ and pressure $p(t, r, z)$ we obtain the following system of equations and initial and boundary conditions:

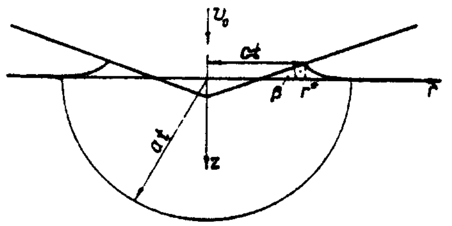

Fig. 1

$$\begin{gathered}\Delta \mathbf{v} = \partial^2 \mathbf{v}/a^2\partial t^2, \quad \Delta p = \partial^2 p/a^2\partial t^2 \quad \text{for } z > 0,\ t > 0 \\ \mathbf{v} = \partial \mathbf{v}/\partial t = 0, \quad p = \partial p/\partial t = 0 \quad \text{for } t = 0 \\ v_z = v_0 \quad \text{for } z = 0,\ 0 \leqslant r < ct,\ p = 0 \quad \text{for } z = 0,\ ct < r\end{gathered} \tag{1.1}$$

where $\mathbf{v}$ and p are related by linearized Euler equations. Furthermore, we stipulate that $p \to 0$ and $\mathbf{v} \to 0$ for $r^2 + z^2 \to \infty$, and that p and $\mathbf{v}$ are integrable in the neighborhood of the boundary of the cone wetted surface at $z = 0$ and $r = ct$, and at the cone vertex at $z = 0$ and $r = 0$, since this is necessary, if a unique solution is to be obtained.

The solution of system (1.1) contains parameter c which is determined by Wagner's method [3], using the kinematic relationship

$$-\int_0^t v_z(\tau, ct, 0)\, d\tau + v_0 t = ct \operatorname{tg} \beta \tag{1.2}$$

between the motion of a particle of the free surface (r^* in Fig. 1) and that of the cone.

2. Relation between axisymmetric and plane problems, and the method of functionally-invariant solutions. A method of reducing axisymmetric problems involving wave equations to plane problems was proposed by V.I. Smirnov and S.L. Sobolev (see Chapter 12 in [7]) by which the solution of the axisymmetric problem is derived by the superposition of solutions of plane problems. We introduce the system of Cartesian coordinates $\xi\eta z$ at angle ω to the z-axis, so that

$$\xi = x\cos\omega + y\sin\omega = r\cos(\varphi-\omega),\quad \eta = -x\sin\omega + y\cos\omega = r\sin(\varphi-\omega)$$

where r, φ and z represent a system of cylindrical coordinates related to the xyz system by formulas: $x = r\cos\varphi$, $y = r\sin\varphi$ and $z = z$.

Let us consider in the system of coordinates $\xi\eta z$ the plane solution of the linearized equations of motion of a perfect fluid, such that the velocity vector v_1 and pressure p_1 are independent of η and which satisfy the wave equation

$$\partial^2 f/\partial\xi^2 + \partial^2 f/\partial z^2 = \partial^2 f/a^2\partial t^2$$

with the velocity vector v_1 lying in the ξz-plane and $\mathrm{v}_1 = \{v_{1\xi},\ v_{1z}\}$. Then the expressions

$$\mathrm{v} = \int_{-\pi}^{\pi} \mathrm{v}_1(t,\xi,z)\,d\omega \quad\text{and}\quad p = \int_{-\pi}^{\pi} p_1(t,\xi,z)\,d\omega \tag{2.1}$$

represent the solution of a certain three-dimensional problem with linearized equations of motion of a perfect fluid, since they are the result of superposition of solutions of equations of motion. In this case functions v and p satisfy the wave equation with three spatial variables x, y and z. For the velocity components v_r, v_z and v_φ and for pressure p we obtain the following expressions (after substituting $\varphi - \omega = \Omega$ and taking into account the periodicity of integrands in (2.1) with respect to Ω):

$$v_r = 2\int_0^\pi v_{1\xi}(t, r\cos\Omega, z)\cos\Omega\,d\Omega,\quad v_z = 2\int_0^\pi v_{1z}(t, r\cos\Omega, z)\,d\Omega$$

$$v_\varphi = 0,\quad p = 2\int_0^\pi p_1(t, r\cos\Omega, z)\,d\Omega \tag{2.2}$$

Hence velocity v and pressure p are independent of φ, i.e. they are solutions of a certain axisymmetric problem with linearized equations of motion of a perfect fluid and satisfy the wave equation

$$\partial^2 f/\partial x^2 + \partial f/dy^2 + \partial^2 f/\partial z^2 = \partial^2 f/a^2\partial t^2$$

It can be shown that formulas (2.2) establish a one-to-one relationship between solutions of plane and axisymmetric problems (*). As in the self-similar problem considered in Sect. 1, v and p are homogeneous functions of coordinates and of the time of zero reading, hence the plane solutions v_1 and p_1 must, also, be homogeneous functions of the time of zero reading, and by the Sobolev-Smirnov method of functionally-invariant solutions [7] can be represented by

$$v_{1\xi}(t,\xi,z) = \operatorname{Re}V(\theta),\qquad v_{1z}(t,\xi,z) = \operatorname{Re}W(\theta)$$
$$p_1(t,\xi,z) = \operatorname{Re}U(\theta) \tag{2.3}$$

where $U(\theta)$, $V(\theta)$ and $W(\theta)$ are analytic functions in region $\operatorname{Im}\theta > 0$, and θ is implicitly determined by the equation

*) B.V.Kostrov. Certain dynamic problems of the mathematical theory of elasticity. Candidate's dissertation, Moscow, 1964.

$$\delta \equiv t - \theta\xi - z\sqrt{a^{-2} - \theta^2} = 0 \qquad (2.4)$$

The branch of the radical is fixed as follows: the θ-plane is cut along the intervals of the real axis $(-\infty, -a^{-1}]$ and $[a^{-1}, +\infty)$ and the radical is assumed positive at $\theta = 0$. Equation (2.4) maps the upper semicircle $z > 0$, $z^2 + \xi^2 < a^2t^2$ of the real half-plane ξz on the upper half-plane of the complex variable θ, the semicircle $z^2 + \xi^2 = a^2t^2$ on segment $[-a^{-1}, a^{-1}]$ of the real axis, and segment $[-at, at]$ on the remaining part of the real axis. According to [7] the plane solution obtained inside the semicircle continuously passes through the arc of circle $z^2 + \xi^2 = a^2t^2$ into the exterior of the semicircle along the half-tangents defined by Eq. (2.4), because θ, as a solution of (2.4), is constant along the latter (it is assumed that for $z > at$ the plane solution is identically zero). The derived plane solution is, thus, the general solution of the wave equation [7].

Substituting expressions (2.3) into (2.2), for the velocity components v_r and v_z, and pressure p in the self-similar axisymmetric problem we obtain the following formulas:

$$v_r = 2\,\mathrm{Re}\int_0^\pi V(\theta)\cos\Omega\, d\Omega, \qquad v_z = 2\,\mathrm{Re}\int_0^\pi W(\theta)\, d\Omega$$

$$p = 2\,\mathrm{Re}\int_0^\pi U(\theta)\, d\Omega \qquad (2.5)$$

where θ is implicitly determined by (2.4) for $\xi = r\cos\Omega$ or, for the chosen branch of the radical $(a^{-2} - \theta^2)^{1/2}$ and half-tangents, explicitly by formulas

$$\theta = \frac{atr\cos\Omega + iz\sqrt{a^2t^2 - z^2 - r^2\cos^2\Omega}}{a(r^2\cos^2\Omega + z^2)} \qquad \text{for } a^2t^2 > r^2\cos^2\Omega + z^2$$

$$\theta = \frac{atr\cos\Omega \pm z\sqrt{z^2 + r^2\cos^2\Omega - a^2t^2}}{a(r^2\cos^2\Omega + z^2)} \qquad \text{for } a^2t^2 < r^2\cos^2\Omega + z^2, \quad z < at \qquad (2.6)$$

The signs plus and minus in the second of formulas (2.6) relate, respectively, to $\cos\Omega < 0$ and $\cos\Omega > 0$, and the radicals in the two formulas are considered to be arithmetic (setting for $z > at$ in formulas (2.2) $\mathbf{v}_1 \equiv 0$ and $p_1 \equiv 0$, we obtain $\mathbf{v} \equiv 0$ and $p \equiv 0$).

Since $\mathbf{v}_1$ and p_1 satisfy the linearized Euler equations

$$\partial v_{1z}/\partial t = -\partial p_1 / \rho\partial z, \qquad \partial v_{1\xi} / \partial t = -\partial p_1 / \rho\partial\xi$$

(where ρ is the density of the unperturbed fluid), functions $U(\theta)$, $V(\theta)$ and $W(\theta)$ are not independent. The Euler equations are, obviously, satisfied when

$$V'(\theta) = U'(\theta)\,\theta / \rho, \qquad W'(\theta) = U'(\theta)\sqrt{a^{-2} - \theta^2} / \rho \qquad (2.7)$$

Thus the solution of the self-similar axisymmetric problem for $\mathbf{v}$ and p is sought in the form (2.5) in which $U(\theta)$, $V(\theta)$ and $W(\theta)$ are regular functions in the region $\mathrm{Im}\,\theta > 0$ and related by formulas (2.7), with θ determined by (2.4) or (2.6).

3. Solution of the problem. For solving the considered axisymmetric problem we use the method proposed in [8] for solving similar problems of elasticity. Differentiating Eqs. (2.5) for v_z and p with respect to t, at $z = 0$ we obtain

$$\frac{\partial p}{\partial t} = 2\,\mathrm{Re}\int_0^\pi \frac{U'(\theta)}{r\cos\Omega}\,d\Omega, \qquad \frac{\partial v_z}{\partial t} = 2\,\mathrm{Re}\int_0^\pi \frac{W'(\theta)}{r\cos\Omega}\,d\Omega \tag{3.1}$$

We introduce into Eqs. (3.1) the new variable ν defined by formula $\theta = \nu^{1/2}$ by slitting plane ν along the positive semiaxis $[0, +\infty)$. The half-plane $\mathrm{Im}\,\theta > 0$ is then imaged in plane ν with slit $[0, +\infty)$. Taking into account Eq. (2.7) which relates $U'(\theta)$ and $W'(\theta)$, from (3.1) we obtain

$$\mathrm{Re}\int_l \frac{F'(\nu)\,d\nu}{\sqrt{\nu-\nu_0}} = \frac{r}{2}\frac{\partial p}{\partial t}, \qquad \mathrm{Re}\int_l \frac{F'(\nu)\sqrt{a^{-2}-\nu}}{\sqrt{\nu-\nu_0}}\,d\nu = \frac{r\rho}{2}\frac{\partial v_z}{\partial t} \tag{3.2}$$

$$\nu_0 = t^2 / r^2, \qquad U(\theta) = U(\nu^{1/2}) \equiv F(\nu)$$

where $F(\nu)$ must be a regular function in the plane ν outside the slit $[0, +\infty)$. For the branch of radical $(a^{-2} - \nu)^{1/2}$ the slit is made along the interval of the positive semiaxis $[a^{-2}, +\infty)$ and the radical is assumed positive for $\nu = 0$; to separate the single-valued branch of radical $(\nu - \nu_0)^{1/2}$ we make the slit $[\nu_0, +\infty)$ and assume the radical to be equal $\pi / 2$ for $\nu = 0$. The contour l (Fig. 2) is obtained as follows. The contour l_0 along which the integration path $[0, \pi]$ of formulas (2.5) for

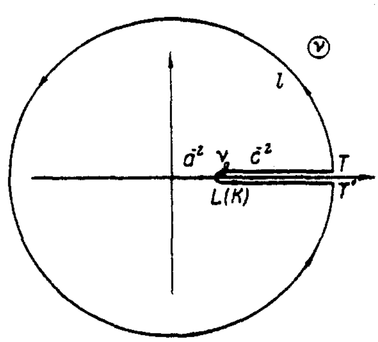

Fig. 2

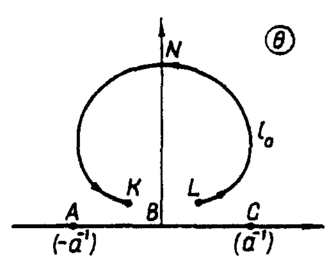

Fig. 3

$z > 0$ passes in the θ-plane is shown in Fig. 3. For $z = 0$ $(z \to +0)$ the ends of contour l_0 (points L and K) lie within segment $[-a^{-1}, a^{-1}]$ and are symmetric about the origin $\theta = 0$. Hence the substitution $\theta = \nu^{1/2}$ converts contour l_0 into contour l which can be represented by a circle of an arbitrary radius R (owing to the analyticity of $F(\nu)$ outside the slit $[0, +\infty)$) and two equal length segments KT' and LT along the lower and upper edges, respectively, of the slit $[0, +\infty)$, with points K and L coinciding with point $\nu = \nu_0$.

To satisfy the initial conditions for Eqs. (3.2) it is necessary that for $\nu_0 < a^{-2}$ the integration contour l can be contracted to a single point, i.e. function $F'(\nu)$ must be analytic outside the slit $[a^{-2}, +\infty)$. In accordance with boundary conditions the expressions for $\partial p / \partial t$ appearing in (3.2) must vanish for $\nu_0 < c^{-2}$, hence $F'(\nu)$ must be a regular function outside the slit $[c^{-2}, +\infty)$. Since the derivative $\partial v_z / \partial t$ in (3.2) must vanish for $\nu_0 > c^{-2}$, the integrand in this expression must be analytic for $\mathrm{Re}\,\nu > \nu_0 > c^{-2}$ and decrease at infinity as $o(\nu^{-1})$ in order that the integral along the circumference vanishes for $R \to \infty$. We can now set

$$F'(\nu) = A(\nu)(c^{-2} - \nu)^{-n}$$

where n is an integer and $A(\nu)$ is an entire analytic function which does not vanish for $\nu = c^{-2}$. The condition of integrability of pressure at the edge of the cone wetted surface implies that $n \leqslant 2$. It will be readily seen from this that $A(\nu)$ must be bounded: $A(\nu) = A \equiv \text{const}$ and $n = 2$. We thus have

$$F(\nu) = A\nu c^2 (c^{-2} - \nu)^{-1} + c_1$$

For $z = 0$, similarly to formulas (3.2), we have

$$p = \sqrt{\nu_0}\,\mathrm{Re}\int_l \frac{F(\nu)\,d\nu}{\nu\sqrt{\nu - \nu_0}}, \qquad v_z = \frac{\sqrt{\nu_0}}{\rho}\mathrm{Re}\int_l \frac{1}{\nu\sqrt{\nu-\nu_0}} \times$$

$$\left[\int_0^\nu F'(\mu)\sqrt{a^{-2} - \mu}\,d\mu + c_2\right] d\nu \tag{3.3}$$

Integration in (3.3) with respect to μ is carried out along the contour which lies on the same side of the real axis as point $\mu = \nu$. It follows from (3.3) that, if the initial conditions are to be satisfied, it is necessary for the integrands in (3.3) to be analytic at point $\nu = 0$, i.e. $c_1 = c_2 = 0$ and, consequently,

$$F(\nu) = A\nu c^2 (c^{-2} - \nu)^{-1}$$

which brings formulas (3.3) to the form (3.4)

$$p = c^2\sqrt{\nu_0}\,\mathrm{Re}\int_l \frac{A\,d\nu}{(c^{-2} - \nu)\sqrt{\nu - \nu_0}}, \qquad v_z = \frac{\sqrt{\nu_0}}{\rho}\mathrm{Re}\int_l \frac{A\,d\nu}{\nu\sqrt{\nu - \nu_0}}\int_0^\nu \frac{\sqrt{a^{-2} - \mu}}{(c^{-2} - \mu)^2}\,d\mu$$

Formulas (3.4) show that p vanishes for $\nu_0 < c^{-2}$. Let us determine constant A from the boundary condition that $v_z = v_0$ for $\nu_0 > c^{-2}$ and $z = 0$. We have

$$\frac{\sqrt{\nu_0}}{\rho}\mathrm{Re}\int_l \frac{A\,d\nu}{\nu\sqrt{\nu - \nu_0}}\int_0^\nu \frac{\sqrt{a^{-2} - \mu}\,d\mu}{(c^{-2} - \mu)^2} = v_0$$

where the integral with respect to μ can be expressed by

$$\int_0^\nu \frac{\sqrt{a^{-2} - \mu}}{(c^2 - \mu)^2}\,d\mu = \int_0^{-\infty} \frac{|\sqrt{a^{-2} - \mu}}{(c^{-2} - \mu)^2}\,d\mu + \int_{\nu\infty}^{\nu} \frac{\sqrt{a^{-2} - \mu}}{(c^{-2} - \mu)^2}\,d\mu = B + F_0(\nu)$$

$$B = -c\,[\gamma^{1/2} + (1 - \gamma)^{-1/2}\arccos(\gamma^{1/2})], \qquad \gamma = c^2 a^{-2}$$

The second integral is integrated along the half-line $\arg\mu = \arg\nu$. Since $F_0(\nu)$ changes its sign at transition through the slit $[c^{-2}, +\infty)$, hence

$$\int_l \frac{F_0(\nu)\,d\nu}{\nu\sqrt{\nu - \nu_0}} = 0 \quad \text{for } \nu_0 > c^{-2}$$

and we obtain $v_0 = 2\pi AB\rho^{-1}$. From which

$$A = -\frac{v_0\rho\sqrt{1-\gamma}}{2\pi c\,[\sqrt{\gamma(1-\gamma)} + \arccos\sqrt{\gamma}]} \tag{3.5}$$

For the distribution of pressure p at the cone wetted surface ($\nu_0 > c^{-2}$) we obtain

$$p = \frac{v_0 \rho c^2 t \sqrt{1-\gamma}}{\sqrt{c^2t^2 - r^2}\,[\sqrt{\gamma(1-\gamma)} + \arccos\sqrt{\gamma}]} \tag{3.6}$$

where p has an integrable singularity at $r = ct$.

To determine the pressure at any point of the half-space $z \geqslant 0$ we substitute the variable $\mu = (a^{-2} - \theta^2)^{1/2}$ into formula (2.5) for p and, using the branch of the radical $(a^{-2} - \theta^2)^{1/2}$, for $a^2t^2 > r^2 + z^2$, we obtain

$$p = 2Ac^2 \mathrm{Re} \int_{l_1} \frac{(za^{-2} - t\mu)\, d\mu}{(c^{-2} - a^{-2} + \mu^2)\,[r^2a^{-2} - t^2 + 2tz\mu - \mu^2(r^2 + z^2)]^{1/2}} \tag{3.7}$$

The contour l_1 in region $\mathrm{Re}\,\mu > 0$ is shown in Fig. 4 and A is defined by formula (3.5).

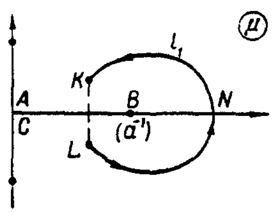

Fig. 4

At points K and L we have, respectively, $\mu = \mu_1$ and $\mu = \mu_2$, where μ_1 and μ_2 are roots of the quadratic trinomial of the radical appearing in (3.7).

To determine the single-valued branch of the radical in (3.7), we make a slit in the μ-plane between points L and K (shown in Fig. 4 by the dash line) and take that branch of the radical whose argument for real μ greater than a^{-1} is equal $\pi/2$. Then, taking into account that along the different edges of the slit the integrand in (3.7) is of the same absolute value but of opposite signs and has two simple poles at points $\mu = \pm i(c^{-2} - a^{-2})^{1/2}$, and using the residue theorem, from (3.7) we finally obtain

$$p = -2\pi Ac^2 \left[t \left(\frac{\sqrt{A_0^2 + B_0^2} - A_0}{2(A_0^2 + B_0^2)} \right)^{1/2} - \frac{z}{a^2\sqrt{c^{-2} - a^{-2}}} \left(\frac{\sqrt{A_0^2 + B_0^2} + A_0}{2(A_0^2 + B_0^2)} \right)^{1/2} \right] \tag{3.8}$$

$$r^2 + z^2 < a^2t^2, \quad A_0 = z^2(c^{-2} - a^{-2}) + c^{-2}r^2 - t^2, \quad B_0 = 2tz((c^{-2} - a^{-2})^{1/2}$$

where the radicals are assumed to be arithmetical. In particular, for $z \to 0$ and $r > ct$ the right-hand side of (3.8) vanishes, while for $z \to 0$ and $r < ct$ it coincides with formula (3.6). Note that for $r^2 + z^2 \to a^2t^2$ formula (3.8) yields zero, while for $c \to a$ from (3.5) we obtain $A \to -v_0\rho/(4\pi a)$, and at the limit (3.8) yields

$$p = \frac{v_0 \rho a^2 t}{2} \frac{a^2t^2 - r^2 - z^2}{(a^2t^2 - r^2)^{3/2}}$$

This formula is the same as the limit formula obtained in the solution of the problem of a blunt cone penetration into a compressible fluid for $c = v_0 \operatorname{ctg}\beta > a$, when $v_0 \operatorname{ctg}\beta \to a$ [6]. In the other limit case $(a \to \infty)$ formula (3.8) yields pressure distribution for the case of incompressible fluid.

The velocity components v_z and v_r in region $r^2 + z^2 < a^2t^2$ can be determined in a similar manner. Note that for $r^2 + z^2 > a^2t^2$ $(z < at)$ functions v_z, v_r and p vanish, since the end-points K and L of the integration contour l_1 in the μ-plane are at one and the same point lying on the real axis segment $[0, a^{-1}]$ in that plane (since in the θ-plane they lie on segment $[-a^{-1}, a^{-1}]$ and are symmetric about point $\theta = 0$). Consequently, by the Cauchy theorem the integrals over the closed contour l_1 vanish.

Let us determine the rate of increase of the radius of the wetted surface c. Formula (3.4) for v_z at the free surface $(v_0 < c^{-2})$ with allowance for (3.5) assumes the form

$$v_z = v_0 \left\{1 - \left(\sqrt{\gamma(1-\gamma)} + \arccos \sqrt{\gamma}\right)^{-1} \left[\arccos \frac{ct}{r} + \frac{ct\,(1-\gamma)}{\sqrt{r^2 - c^2t^2}}\right]\right\} H\,(at - r)$$

where $H\,(at - r)$ is the Heaviside unit function. Substituting this expression for v_z into (1.2) and integrating, we finally obtain the expression

$$\gamma + \gamma^{1/2}\,(1-\gamma)^{-\frac{1}{2}} \arccos\,(\gamma^{1/2}) = 2v_0a^{-1} \operatorname{ctg} \beta \qquad (3.9)$$

which defines the relationship between $v_0a^{-1} \operatorname{ctg} \beta$ and γ. The numerical solution of Eq. (3.9) is presented in Fig. 5 in the form of a curve showing the dependence of $v_0^{-1} c \operatorname{tg} \beta$ on $v_0a^{-1} \operatorname{ctg} \beta$. It will be seen that c depends on a for constant $v_0 \operatorname{ctg} \beta$ and $c \to 4\pi^{-1}v_0 \operatorname{ctg} \beta$ for $v_0a^{-1} \operatorname{ctg} \beta \to 0$ (the case of an incompressible fluid). When $v_0a^{-1} \operatorname{ctg} \beta \to 1$, $c \to v_0 \operatorname{ctg} \beta \to a$ and the fluid free surface around the cone remains undisturbed, which corresponds to the physical pattern of flow.

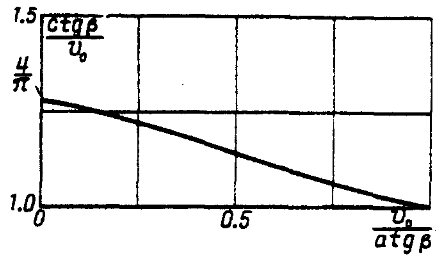

Fig. 5

Reverting to formula (3.6), we note that, when parameter c is specified and is independent of a, the pressure determined by that formula depends on a, contrary to the other limit case, i.e. of penetration of a narrow cone at subsonic velocity into a compressible fluid, in which the pressure on the cone is independent of a [6].

The formula for the pressure on the surface of a blunt cone penetrating into an incompressible fluid [2], obtained from (3.6) with $a \to \infty$, is

$$p = \frac{2}{\pi}\, v_0 \rho c \left[1 - \left(\frac{r}{ct}\right)^2\right]^{-1/2} \qquad (3.10)$$

where ct, as in (3.6), is the radius of the cone wetted surface. Formula (3.6) differs from (3.10) by the presence of the coefficient $2^{-1}\pi\,[\gamma^{1/2} + (1-\gamma)^{-1/2} \arccos\,(\gamma^{1/2})]^{-1}$, which varies from $\pi / 4$ to unity, hence for the same values of v_0, r, t and c the pressure exerted by a compressible fluid is lower than that produced by an incompressible fluid.

When c is specified in formula (3.6) by Eq. (3.9), we obtain for the pressure on the blunt cone surface the formula

$$p = \frac{1}{2}\, \rho c^2 \operatorname{tg} \beta \left[1 - \left(\frac{r}{ct}\right)^2\right]^{-1/2} \qquad (3.11)$$

If in the case of a compressible fluid the rise of its free surface is neglected, then $c = v_0 \operatorname{ctg} \beta$, and formula (3.6) assumes the form

$$p = \frac{v_0^2\rho}{\operatorname{tg}\beta}\left[1 - \left(\frac{r \operatorname{tg} \beta}{v_0 t}\right)^2\right]^{-1/2} \left\{\frac{v_0}{a \operatorname{tg} \beta} + \left[1 - \left(\frac{v_0}{a \operatorname{tg} \beta}\right)^2\right]^{-1/2} \arccos \frac{v_0}{a \operatorname{tg} \beta}\right\}^{-1} \qquad (3.12)$$

If in the case of an incompressible fluid the rise of its free surface is taken into account, i.e. according to [3] for $c = 4v_0\pi^{-1} \operatorname{ctg} \beta$, the distribution of pressure at the cone is given by

$$p = \frac{8v_0^2\rho}{\pi^2 \operatorname{tg} \beta}\left[1 - \left(\frac{\pi r \operatorname{tg} \beta}{4v_0 t}\right)^2\right]^{-1/2} \qquad (3.13)$$

and, if the fluid rise is neglected, it is determined by formula (3.10), in which $c = v_0 \operatorname{ctg} \beta$ is to be set, we have

$$p = \frac{2v_0^2\rho}{\pi \operatorname{tg}\beta}\left[1-\left(\frac{r \operatorname{tg}\beta}{v_0 t}\right)^2\right]^{-1/2} \tag{3.14}$$

Let us determine the forces acting on a blunt cone surface at one and the same instant t for the four pressures calculated by formulas (3.11) – (3.14), and denote the results by F_n ($n = 1, 2, 3, 4$), respectively. Relating these to F_4, we obtain

$$\frac{F_1}{F_4} = \frac{\pi}{4}\left(\frac{c \operatorname{tg}\beta}{v_0}\right)^4, \quad \frac{F_2}{F_4} = \frac{\pi}{2}\left\{\frac{v_0}{a \operatorname{tg}\beta} + \left[1-\left(\frac{v_0}{a \operatorname{tg}\beta}\right)^2\right]^{-1/2} \arccos\frac{v_0}{a \operatorname{tg}\beta}\right\}^{-1}$$

$$F_3 / F_4 = (4/\pi)^3, \qquad F_4 / F_4 = 1$$

Curves of the dependence of F_n/F_4 on $v_0 a^{-1} \operatorname{ctg}\beta$ are shown in Fig. 6, where they are denoted by *1*—*4*, respectively. It should be noted that the difference of hydrodynamic force F_n acting on the surface of the penetrating cone in these four cases is due not only to the difference of pressures but, also, to the differences of the cone wetted surface. These curves show that the behavior of this force is most accurately and physically correctly defined by curve *1* which for $v_0 a^{-1} \operatorname{ctg}\beta \to 0$ corresponds to the case of an incompressible fluid, curve *3* in which the rise of the fluid free surface is taken into account, while for $v_0 a^{-1} \operatorname{ctg}\beta \to 1$, when $c \to a$, the point $(1, \pi/4)$ of this curve yields the similar limit value for the case of $c = v_0 \operatorname{ctg}\beta > a$ for $v_0 \operatorname{ctg}\beta \to a$. In the latter limit case the theory of incompressible fluid, with the rise of fluid taken into account (curve *3*) yields a value which is approximately 2.6 times higher than the limit value shown by curve *1*. Hence curve *1* must be used for the correct determination of hydrodynamic forces within the range $0 < v_0 a^{-1} \operatorname{ctg}\beta < 1$.

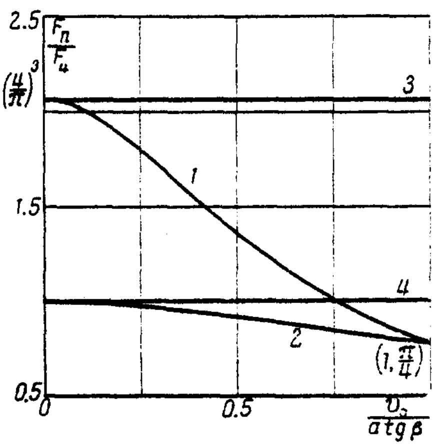

Fig. 6

We note in conclusion that, when the cone is not infinite and the radius of its base is equal b, the problem is self-similar only so far as its wetted surface does not reach its base, where $r = b$, i.e. up to the instant $t_0 = b / c$. It can be expected that, as in the case of an incompressible fluid [3], the force acting at that instant on the cone penetrating at constant velocity into the fluid reaches its maximum $F_1(t_0)$. Let us compare this maximum force with the maximum force $F_3(t_1)$ determined for the same problem by the theory of incompressible fluid [3], with the rise of fluid taken into account in both cases. Considering that in the case of incompressible fluid the maximum force is reached at instant $t_1 = \pi b \operatorname{tg}\beta / (4v_0)$, using Eqs. (3.11) and (3.13), we obtain

$$\frac{F_3(t_1) - F_1(t_0)}{F_1(t_0)} = \left(\frac{4v_0}{\pi c \operatorname{tg}\beta}\right)^2 - 1$$

where c is defined by Eq. (3.9). The right-hand side of this expression is a monotonically increasing function of parameter $v_0 a^{-1} \operatorname{ctg}\beta$ which for $v_0 a^{-1} \operatorname{ctg}\beta \to 0$ tends to vanish and for $v_0 a^{-1} \operatorname{ctg}\beta \to 1$ reaches the value $16\pi^{-2} - 1 \approx 0{,}62$.

Thus for v_0 ctg $\beta = a$ the maximum force derived by the theory of incompressible fluid exceeds by 62 % that calculated for a compressible fluid.

The author thanks E. I. Grigoliuk and S. S. Grigorian for discussing this problem.

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Translated by J. J. D.

UDC 532. 517. 4

ASYMMETRIC MECHANICS OF TURBULENT FLOWS. ENERGY AND ENTROPY

PMM Vol. 37, №1, 1973, pp. 94-105

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(Received May 22, 1972)

Averaged equations of motion of a turbulized fluid in the presence of a preferred orientation of turbulent vortices were constructed in [1]. By taking account of an additional kinematic variable, the angular velocity of vortex self-rotation, the system of equations in [1] differs from the earlier theory of Mattioli [2].

The equations from [1] are supplemented herein by a turbulent energy balance equation in which the work of the moment stresses and the antisymmetric component of the Reynolds stress tensor is taken into account. It is shown that the inner energy determined by turbulization of the fluid depends on the root-mean-square values of the translational pulsation velocities and the anglular vortex velocities. The entropy and "temperature" of turbulization are introduced; the